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Quantum Implication

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This article is to discuss the several implications introduced in quantum mechanics. This discussion will be from a rigorous logical point of view.

The main purpose of this article is to discuss the several implications introduced in quantum mechanics. The reader should consult Pavičić [9] for a discussion of the implications arising from quantum mechanics.

To our knowledge the quantum implication has not been discussed from a rigorous logical point of view. As is well known, if we have a certain logic (e.g., classical intuitionistic, linear, etc.), we want to introduce an implication in order that we can in fact make *deductions* (or inferences) in a proper way. What we mean by this is simply that *some* kind of *deduction theorem* must hold in a logic. Since the appearance of categorical logic, we can in fact formulate logic from an algebraic point of view. Perhaps one of the first persons who realized this idea was J. Lambek. See, for instance, ref. 4, where he discusses how to see deductive systems as categories.

We shall see that the only implication satisfying a proper deduction theorem is the so-called Sasaki hook. Our approach will be categorical, but the reader not familiar with category theory or categorical logic can read this paper without difficulty, since all our proofs and statements use only basic knowledge of the theory of posets, which in fact is the main framework for quantum logic.

Quantum logic is a very interesting logic. The main point of this logic is that it gives a *real* example of how *new* connectives arise from concrete mathematics. Since the birth of *linear logic* (see ref. 1 for details) this

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phenomenon has been common in categorical logic; in fact, there is a deep similarity of these two logics; we consider this further in ref. 7.

Recent work on *D*-posets introduced in ref. 3 is also related to this approach; the interested reader should see ref. 7 for more details.

1. A CHARACTERIZATION OF THE SASAKI ARROW

Throught this work, L will denote an *orthomodular lattice*, that is, a lattice $(L, \wedge, \vee, 0, 1)$ with a *unary* operation $\perp: L \to L$ satisfying the following:

(i) $a \leq b \Rightarrow b^{\perp} \leq a^{\perp}$.
(ii) $a^{\perp} \vee a = 1$, $a^{\perp} \wedge a = 0$.
(iii) Whenever $a \leq b$, then $b = a \vee (a^{\perp} \wedge b)$.

These lattices have traditionally been associated with quantum logic since the propositions for a quantum system correspond to closed subspaces of a Hilbert space and these constitute an orthomodular lattice.

In order to turn an orthomodular lattice L into a logical system, an appropriate implication has to be defined. We will see that this logical system differs completely from such well-known systems as classical logic and intuitionistic logic. In ref. 9 several implications are discussed for an orthomodular lattice, it is claimed that no implication can be defined since the deduction theorem does not hold. The point is that one can define an implication in such way that the deduction theorem holds, but one must define a new conjunction operator. In categorical terms, the deduction theorem says that the implication has a left adjoint (viewed, of course, as a functor). Therefore we will see that the Sasaki arrow has a left adjoint and is the only implication that satisfies this property if one admits three basic implicative criteria. Now, these three criteria come from quantum mechanics; see ref. 8 for a discussion of these implicative criteria. In order for this paper to be self-contained, we first define the implications discussed in ref. 9.

Definition 1.1. Let L be an orthomodular lattice. We introduce the following operations of implication $(a, b, \in L)$:

1. $a \to_1 b = a^{\perp} \vee b$.
2. $a \to_2 b = a^{\perp} \vee (a \wedge b)$.
3. $a \to_3 b = b \vee (a^{\perp} \wedge b^{\perp})$.
4. $a \to_4 b = (a \wedge b) \vee (a^{\perp} \wedge b) \vee (a^{\perp} \wedge b^{\perp})$.
5. $a \to_5 b = (a \wedge b) \vee (a^{\perp} \wedge b) \vee [(a^{\perp} \vee b^{\perp}) \wedge b^{\perp}]$.
6. $a \to_6 b = (a^{\perp} \wedge b^{\perp}) \vee (a^{\perp} \wedge b) \vee [(a^{\perp} \vee b) \wedge a]$.

Remark 1. The first implication is the classical one, the second is the Sasaki arrow, and the rest were defined in order to provide an appropriate implication for orthomodular lattices. In ref. 5 we noted that if we take $a \rightarrow_2$ —, then it has a left adjoint given by

$$—\&a: \quad L \rightarrow L$$

where for any b in L, $b\&a = (b \vee a^{\perp}) \wedge a$. Notice that $b\&a$ is greater than $a \wedge b$, so we can think of & as a generalization of the well-known connective $\wedge$. An interesting fact of this operation & is that it is not commutative and associative unless one has a Boolean algebra. Moreover, $a\&$— is not a functor, only when one has again a Boolean algebra. In fact, all of these three conditions are equivalent in the sense if that one wants to turn an orthomodular lattice into a Boolean algebra, then one only needs that one of these conditions holds; see refs. 5 and 6 for details.

One of the interesting properties that all these implications satisfy is the following: if two elements a, b in L are such that $a\&b = a \wedge b$, then $a \rightarrow_i b = a^{\perp} \vee b$ for $i \in \{2, \ldots, 6\}$, as the reader can check easily. This condition is equivalent to the well-known assertion that two elements a, b in an orthomodular L are compatible (denoted by aCb) iff the set $\{a, b\}$ generates a Boolean subalgebra of L. See ref. 8 for more comments on compatibility. We believe that the first condition is more logical and categorical since we can express it in equational form.

Furthemore, it is desirable that the usual connection between implication and the order relation holds; i.e., if a, b are two arbitrary elements of L, then the following is true:

$$a \leq b \quad \text{iff} \quad a \rightarrow b = 1$$

In order to give our characterization of the Sasaki arrow, we begin with the following result.

Lemma 1.2. Let $\rightarrow$: $L \times L \rightarrow L$ and $\oplus$: $L \times L \rightarrow L$ be two binary operations satisfying the following condition: If a, b, c are arbitrary elements of L, then

$$a \oplus b \leq c \quad \text{iff} \quad a \leq b \rightarrow c$$

Then the function $x \rightarrow$ —: $L \rightarrow L$ is order preserving for all x in L.

Proof. Suppose a, b in L are such that $a \leq b$. Then $x \rightarrow a \leq x \rightarrow b$ iff $(x \rightarrow a) \oplus x \leq b$. Now, clearly, $(x \rightarrow a) \oplus x \leq a$ and by hypothesis we get $(x \rightarrow a) \oplus x \leq a \leq b$. Hence, $x \rightarrow a \leq x \rightarrow b$ and $x \rightarrow$ — is order preserving. ■

Remark 2. Notice that this lemma holds for any poset. Moreover, we conclude that there is no way of defining an implication in an orthomodular lattice in such a way that the usual deduction theorem holds, that is, whenever the following condition holds. Given a, b, c in L, then

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \rightarrow c$$

Since $a \rightarrow —$ is order preserving, it would be a right adjoint to $— \wedge a$ for all a in L, implying that L is a distributive lattice (see, for instance, ref. 2), but an orthomodular lattice L cannot be distributive unless L is a Boolean algebra.

The next lemma is important in order to give our characterization of the Sasaki arrow. This lemma holds for any lattice with finite meets.

Lemma 1.3. Let $\rightarrow: L \times L \rightarrow L$ be a binary operation satisfying the following conditions. Given a, b, c in L, then:

(i) $a \leq b$ iff $a \rightarrow b = 1$.
(ii) There exists a binary operation $\oplus: L \times L \rightarrow L$ such that $a \oplus b \leq c$ iff $a \leq b \rightarrow c$.

Then $a \rightarrow (a \wedge b) = a \rightarrow b$.

Proof. By condition (ii) we know $a \rightarrow b \leq a \rightarrow b$ iff $(a \rightarrow b) \oplus a \leq b$. Moreover, by condition (i) $a \rightarrow b \leq a \rightarrow a = 1$ iff $(a \rightarrow b) \oplus a \leq a$; hence, $(a \rightarrow b) \oplus a \leq a \vee b$. On the other hand, $a \rightarrow (a \vee b) \leq a \rightarrow b$ since by Lemma 1.2, $a \rightarrow (—)$ is a morphism of posets. So, we obtain $a \rightarrow (a \wedge b) = a \rightarrow b$, which is the desired result. ∎

Using this lemma, we can prove that the Sasaki arrow is characterized by the following result.

Theorem 1.4. Let $\rightarrow: L \times L \rightarrow L$ be a binary operation satisfying the following conditions. Given a, b, c arbitrary elements of L, we have:

(i) $a \leq b$ iff $a \rightarrow b = 1$.
(ii) There exists a binary operation $\oplus: L \times L \rightarrow L$ such that $a \oplus b \leq c$ iff $a \leq b \rightarrow c$ holds.
(iii) Whenever a, b are two compatible elements of L, then $a \rightarrow b = a^{\perp} \vee b$. Hence $\rightarrow$ is the Sasaki arrow, i.e., $a \rightarrow b = a^{\perp} \vee (a \wedge b)$.

Proof. By Lemma 1.3, $a \rightarrow (a \wedge b) = a \rightarrow b$. Now by condition (iii), since $a \wedge b$ is compatible with a, we have $a \rightarrow (a \wedge b) = a^{\perp} \vee (a \wedge b)$, which is what we want. ∎

Remark 3. Observe that none of the implications except, of course, for $j = 2$ listed in Definition 1.1 satisfies condition (ii) of the last theorem. However, any of the implications satisfies conditions (i) and (iii). As we claimed, the pair $(\rightarrow_2, \&)$ satisfies the deduction theorem, thinking of & as a new logical conjunction. We note that property (i) can be weakened to $a \rightarrow a = 1$ for a in L in Lemma 1.3 and the last theorem.

We close this paper with some comments. The three conditions stated on Theorem 1.4 are natural and important; clearly, we get a noncommutative, nonassociative operation. However, ($\to_2$, &) is a real logic in the sense that a deduction theorem holds. Hence, one can make deductions in the usual way. Moreover, if we take the closed subspaces of a Hilbert space, we can in fact give a physical interpretation of the conjunction &. See ref. 5 for details.

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